

# Conserved Quantities for Polyhomogeneous Space-Times.

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## Abstract

The existence of conserved quantities with a structure similar to the Newman-Penrose quantities in a polyhomogeneous space-time is addressed. The most general form for the initial data formally consistent with the polyhomogeneous setting is found. The subsequent study is done for those polyhomogeneous space-times where the leading term of the shear  $\sigma$  contains no logarithmic terms. It is found that for these space-times the original NP quantities cease to be constants, but it is still possible to construct a set of other 10 quantities that are constant. From these quantities it is possible to obtain as a particular case a conserved quantity found by Chruściel et al.

## 1 Introduction.

In a classical article by Newman & Penrose [4](see also [5], and [12] for a different treatment using the Bondi-Sachs metric) the existence of ten conserved quantities (NP quantities) for the gravitational field was established in a setting that assumed asymptotic flatness. Recently Chruściel, MacCallum

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& Singleton [13] have considered “polyhomogeneous” space-times as an adequate setting for discussing the Bondi-Sachs characteristic initial value problem, showing that polyhomogeneity at null infinity ( $\mathfrak{I}$ ) is formally consistent with the Einstein field equations. They found that for the axisymmetric case and assuming what they call the “minimal sequence” for the polyhomogeneity of the initial data there is a conserved quantity that resembles those of Newman & Penrose. However, the general case (no symmetries, and arbitrary polyhomogeneity) was left as an open question. The objective of this article is to discuss the existence of the analogous of the Newman-Penrose conserved quantities for a broad family of polyhomogeneous space-times.

The organization of this article is as follows: in section 2 the coordinate system and the null tetrad used in this work are defined. In section 3 some remarks concerning the asymptotic characteristic initial value problem are made; the most general form for the logarithmic terms of the Weyl tensor formally consistent with the field equations is investigated. Once the general form is found, we restrict our study to those polyhomogeneous space-times where the leading term of the shear ( $\sigma$ ) contains no logarithms. A possible physical interpretation of these space-times is discussed. In section 4 the existence of conserved quantities similar in structure to the NP quantities is discussed. It is found that for the space-times under study the original NP quantities cease to be constant, but nevertheless it is possible to construct some other constants (10 in general). The relation with a constant quantity found by Chrusciel et al. for the axisymmetric case is shown. Finally in section 5 some concluding remarks are made. There are also 2 appendices. In appendix A there are two tables showing the polyhomogeneous behaviour of the components of the Weyl tensor, spin coefficients and tetrad functions for the general polyhomogeneous case, and the case where  $\sigma$  contains no logarithmic terms. Some relations obtained from the expansions of the Bianchi identities are also listed. In appendix B some useful properties of the differential operator  $\mathfrak{D}$  are listed for quick reference.

## 2 Coordinate system, null tetrad & the Newman-Penrose formalism.

When studying the asymptotics of the Einstein field equations, a well suited coordinate system can be constructed. The construction shown here is fairly

standard and a good reference is J. Stewart's book [1]. As mentioned in the introduction, the NP formalism will be used, and we will refer to the field equations, commutation relations and Bianchi identities as they are labeled in reference [1].

The first step is to introduce a family of null hypersurfaces in the Riemannian manifold. We can use a parameter  $u$  to label the hypersurfaces of the family by  $u = \text{const.}$ ; this defines a scalar field. As usual, the first tetrad vector will be chosen to be orthogonal to the null hypersurfaces,

$$\mathbf{l} = \mathbf{d}u. \quad (1)$$

Since the hypersurfaces are null, the vectors  $l^\mu$  will be tangent to a family of curves on the hypersurfaces  $\gamma_u$  (the generators of the hypersurfaces). These curves are null geodesics,

$$\nabla^{\rightarrow} \overrightarrow{l} = 0. \quad (2)$$

On each generator  $\gamma_u$  we can choose an arbitrary affine parameter  $r$ . Each null hypersurface  $u = \text{const}$  will intersect  $\mathfrak{I}$  in a cut  $S_u$ . On any cut it is possible to introduce arbitrary coordinates  $x^i$  ( $i = 2, 3$ ). The coordinates  $x^i$  are propagated by demanding  $x^i = \text{const.}$  on the generators of  $\mathfrak{I}$  and on the generators of the null hypersurfaces  $\gamma_u$ . In this way we have defined a coordinate system  $(x^0, x^1, x^2, x^3) = (u, r, x^2, x^3)$  on a neighborhood of  $\mathfrak{I}$ . There is still some freedom remaining in the coordinate system just defined, consisting of: a relabeling of the null hypersurfaces; a different choice of the coordinates  $x^i$  on the cuts  $S_u$ ; and a freedom of the scaling and origin of  $r$  (remember it is an affine parameter).

The freedom on the scaling of  $r$  can be used to set

$$\overrightarrow{l} = \frac{\partial}{\partial r}. \quad (3)$$

Together with  $\overrightarrow{l}$  it is possible to define another null vector  $\overrightarrow{n}$  normalized by  $\overrightarrow{l} \cdot \overrightarrow{n} = 1$ . The surfaces  $u = \text{const.}$  and  $r = \text{const.}$  will be denoted by  $S_{u,r}$ . The last two vectors of the tetrad  $\overrightarrow{m}$  and  $\overrightarrow{\bar{m}}$  are chosen to span  $T(S_{u,r})$  (surface forming). The freedom left in this choice is a boost and a spin.

From the construction we get that the components of the tetrad can be written as,

$$l^\mu = \delta_1^\mu, \quad (4)$$

$$n^\mu = \delta_0^\mu + Q\delta_1^\mu + C^j\delta_j^\mu, \quad (5)$$

$$m^\mu = \xi^i\delta_i^\mu, \quad (6)$$

where  $Q$ ,  $C^i$ ,  $\xi^i$  ( $i = 2, 3$ ) are complex functions of the coordinates. Because  $\vec{m}$  &  $\overrightarrow{\vec{m}}$  span  $T(S_{u,r})$  then,

$$\begin{aligned} \xi^i\xi_i &= 0, \\ \xi^i\overline{\xi}_i &= -1 \text{ (normalization).} \end{aligned} \quad (7)$$

Applying the commutators to  $x^i$  we get that:

$$\begin{aligned} \kappa &= \epsilon = 0, \\ \tau &= \overline{\alpha} + \beta, \\ \mu &= \overline{\mu}, \\ \rho &= \overline{\rho}. \end{aligned} \quad (8)$$

### 3 Initial data for the asymptotic characteristic initial value problem.

The NP quantities arise from considering the asymptotic characteristic initial value problem for the vacuum Einstein field equations [4], [5]. The coordinate system constructed before is well suited for the study of this problem because with it the NP field equations form a hierarchy of differential equations that is reasonably easy to solve under certain assumptions. So, it will be necessary to make some remarks on the kind of initial data required to have the problem well posed.

Kánnár [15] has proved an existence and uniqueness theorem for the asymptotic characteristic initial value problem for the vacuum field equations in the case of  $C^\infty$  initial data. The initial data is given on an incoming null hypersurface  $\mathcal{N}$  and on part of the past null infinity  $\mathfrak{I}^-$  (or of the future null infinity  $\mathfrak{I}^+$ ) which intersect in a two-dimensional space-like surface  $\mathcal{Z}$  diffeomorphic to  $S^2$ .

The  $C^\infty$  initial data that assures existence of a unique solution in a neighborhood of  $\mathcal{Z} = \mathfrak{I}^- \cap \mathcal{N}$  (or  $\mathfrak{I}^+ \cap \mathcal{N}$ ) is given by:

$$\begin{aligned} & \hat{\sigma} \text{ on } \mathfrak{I}^- \text{ ( or } \mathfrak{I}^+), \\ & \hat{\Psi}_0 \text{ on } \mathcal{N}, \\ & \text{and } \hat{\Psi}_1, \hat{\Psi}_2 + \hat{\bar{\Psi}}_2 = 2 \operatorname{Re} \Psi_2, \xi^i \text{ on } \mathcal{Z}, \end{aligned} \quad (9)$$

where the hatted quantities belong to the unphysical space-time. This is not the only possible choice of initial data that fits Kánnár's theorem, but it is the one that is generally used when trying to solve the field equations with asymptotic expansions. We have to point out here that Kánnár's theorem works only for  $C^\infty$  initial data. So far, we still do not have an existence/uniqueness theorem for polyhomogeneous initial data; however Winicour [17] has constructed some space-times that possess a logarithmic behaviour similar to the one discussed in this article.

In the Bondi-Sachs treatment the initial data on  $\mathcal{N}$  is contained in the metric functions  $\gamma$  and  $\delta$ . So the information in  $\gamma$  and  $\delta$  is coded in  $\Psi_0$  when working in the NP formalism. In reference [13] it was proved that if  $\gamma$  and  $\delta$  are polyhomogeneous functions of  $r$ , then the remaining metric functions that are obtained from the initial data and the Einstein equations are also polyhomogeneous. Due to the physical equivalence of the Bondi and the Newman-Penrose treatment we have that if  $\Psi_0$  is polyhomogeneous in  $r$  then the NP spin coefficients that are calculated from this initial data will be polyhomogeneous as well as the remaining components of the Weyl tensor and the tetrad functions.

It can also be shown [13] that given a sequence  $\{N_i\}_{i=0}^\infty$  that defines the form of a certain polyhomogeneous function  $f$  through

$$f = \sum_{i=1}^{\infty} \sum_{j=0}^{N_i} f_{ij} r^{-i} \ln^j r, \quad (10)$$

where the  $f_{ij}$  are some functions of  $(u, x^i)$ , then there exists another sequence  $\{\tilde{N}_i\}_{i=0}^\infty$  satisfying  $N_i \leq \tilde{N}_i$  which defines the form of the initial data  $\gamma$  &  $\delta$  that is compatible with the field equations. This means that the polyhomogeneous form of  $\gamma$  &  $\delta$ , or  $\Psi_0$  in our case, is not arbitrary, but there are heavy mathematical restrictions on the form of the initial data. In particular we cannot put arbitrary powers of  $\ln$  in the leading terms of the initial data.

Because of the asymptotic character of our analysis we will not be concerned with questions of convergence. We can always truncate the series to an adequate order  $M$  in the powers of  $1/r$ . For our purposes  $M = 6$  will be enough.

Here arises the question of which is the most general form of  $\Psi_0$  that is compatible with the field equations. To answer this question we will require some preliminary results.

Let,

$$g = \sum_{i=1}^6 g_i(z) r^{-i} + \dots , \quad (11)$$

$$h = \sum_{i=1}^6 h_i(z) r^{-i} + \dots , \quad (12)$$

where  $g_i(z)$  and  $h_i(z)$  are polynomials in  $z = \ln r$  be two polyhomogeneous functions. It can be shown that,

$$hg = \sum_{i=2}^6 \sum_{k=1}^{i-1} h_k(z) g_{i-k}(z) r^{-i} + (\text{terms with higher powers of } 1/r), \quad (13)$$

and that,

$$\begin{aligned} \frac{\partial}{\partial r} h &= \sum_{i=2}^6 (h'_{i-1}(z) - (i-1) h_{i-1}(z)) r^{-i} \\ &\quad + (\text{terms with higher powers of } 1/r), \end{aligned} \quad (14)$$

where the apostrophe ' denotes differentiation with respect to  $z$ .

Now, the most general form for  $\Psi_0$  in the polyhomogeneous setting is

$$\Psi_0 = \sum_{i=1}^6 \Psi_0^i(z) r^{-i} + \dots . \quad (15)$$

Let  $N$  be the maximum of the degrees of the polynomials

$$\Psi_0^i(z) = \Psi_0^{i,N_i} z^{N_i} + \Psi_0^{i,N_i-1} z^{N_i-1} + \dots + \Psi_0^{i,0}, \quad (16)$$

$i = 1 \dots 6$ , where  $\Psi_0^{i,j}$  depend only on  $(u, \theta, \varphi)$ .

To determine the restrictions on  $\Psi_0$  we will use the first two field equations in the NP hierarchy, namely equations (a) & (b) of [1],

$$D\rho = \rho^2 + \sigma\bar{\sigma}, \quad (17)$$

$$D\sigma = 2\rho\sigma + \Psi_0. \quad (18)$$

We will take  $\rho$  &  $\sigma$  to be in principle of the same form as  $\Psi_0$ .

$$\rho = \sum_{i=1}^6 \rho_i r^{-i} + \dots, \quad (19)$$

$$\sigma = \sum_{i=1}^6 \sigma_i r^{-i} + \dots. \quad (20)$$

Substitution into (a) & (b) yields the following useful relations,

$$\rho'_{i-1} - (i-1)\rho_{i-1} = \sum_{k=1}^{i-1} (\rho_k \rho_{i-k} + \sigma_k \bar{\sigma}_{i-k}), \quad (21)$$

and

$$\sigma'_{i-1} - (i-1)\sigma_{i-1} = 2 \sum_{k=1}^{i-1} \rho_k \sigma_{i-k} + \Psi_0^i, \quad (22)$$

for  $i = 2, 3, \dots$ . Using these formulae, after a long but not difficult analysis it is possible to show that the most general form for a polyhomogeneous  $\Psi_0$  formally consistent with (17) and (18) is

$$\Psi_0 = \Psi_0^3[\|N/2\|-1]r^{-3} + \Psi_0^4[N-1]r^{-4} + \Psi_0^5[N]r^{-5} + \Psi_0^6[N]r^{-6} \ln^i r + \dots, \quad (23)$$

where the quantities in square brackets are the degrees of the different polynomials in  $z = \ln r$ . Here  $\|N/2\|$  denotes the integer part of  $N/2$  (i.e.  $\|1/2\| = 0$ ,  $\|1\| = 1$ ,  $\|3/2\| = 1\dots$ ).

As a by-product we also obtain the form for  $\rho$ , and  $\sigma$ ,

$$\rho = -r^{-1} + \rho_3[N-1]r^{-3} + \dots \quad (24)$$

$$\sigma = \sigma_2[\|N/2\|]r^{-2} + \sigma_3[N-1]r^{-3} + \dots. \quad (25)$$

Using the same technique we can obtain the leading terms for the remaining spin coefficients, components of the Weyl tensor and tetrad functions. The results are listed as a table in Appendix A. From that table we read

$$\Psi_1 = \Psi_1^3[\|N/2\| - 1]r^{-3} + \Psi_1^4[N]r^{-4} + \dots, \quad (26)$$

$$\Psi_2 = \Psi_2^3[\|N/2\|]r^{-3} + \Psi_2^4[N]r^{-4} + \dots, \quad (27)$$

$$\Psi_3 = \Psi_3^2[0]r^{-2} + \Psi_2^3[\|N/2\|]r^{-3} + \dots, \quad (28)$$

$$\Psi_4 = \Psi_4^1[0]r^{-1} + \Psi_4^2[0]r^{-2} + \dots. \quad (29)$$

Note that (23) guarantees that  $\Psi_n \rightarrow 0$  when  $r \rightarrow \infty$  in agreement with the results of Couch & Torrence [14], and the modified peeling behaviour of the Riemann tensor of Chruściel et al. [13].

Note that in this setting the peeling theorem is no longer valid. The introduction of coefficients with lower powers in  $1/r$  ( $\Psi_0^3$  and  $\Psi_0^4$ ) translates into stronger incoming radiation fields than those allowed by the so called *outgoing radiation condition*. It was believed that this condition ensured the non existence of incoming radiation of infinite duration. In fact it can be showed that the  $\Psi_0^4$  coefficient gives rise to the  $1/r^2$  missing term in the original Bondi-Sachs treatment. From the way these coefficients relate with the shear  $\sigma$  (see the appendix A, and in particular equations (56) and (62)) we can think of them as a contribution to the shear from the incoming field. For a general polyhomogeneous space-time this shear will dominate over the shear coming from the news function  $\dot{\sigma}_{2,0}$  (i.e. shear associated with the outgoing radiation), because  $r^{-2} \ln^k r$  ( $k > 0$ ) falls to 0 in a slower way than  $r^{-2}$  when  $r \rightarrow \infty$ . Of course this interpretation is not completely rigorous, and may be regarded only as a guideline.

This picture is somehow disturbing from the physical point of view, because in the study of isolated gravitationally radiating systems we are mainly interested in the outgoing radiation; so we would not like not to have its effects overshadowed by the incoming radiation. We will demand that  $\hat{\sigma}$  is finite on  $\mathfrak{I}$ , this is  $\sigma_2 = \sigma_{2,0}(\theta, \varphi)$  to be a polynomial of degree 0 in  $z = \ln r$  in order to obtain space-times where the main origin of the shear is the outgoing radiation. This fixes the form of the components of the Weyl tensor to be:

$$\Psi_0 = \Psi_0^4[N - 1]r^{-4} + \Psi_0^5[N]r^{-5} + \Psi_0^6[N]r^{-6} \ln^i r + \dots, \quad (30)$$

$$\Psi_1 = \Psi_1^4[N]r^{-4} + \dots, \quad (31)$$

$$\Psi_2 = \Psi_2^3[0]r^{-3} + \Psi_2^4[N]r^{-4} + \dots, \quad (32)$$

$$\Psi_3 = \Psi_3^2[0]r^{-2} + \Psi_2^3[0]r^{-3} + \dots, \quad (33)$$

$$\Psi_4 = \Psi_4^1[0]r^{-1} + \Psi_4^2[0]r^{-2} + \dots. \quad (34)$$

We observe from the formulae (86), (91), and (93) that  $\Psi_0^{4,N-1}$ ,  $\Psi_1^{4,N}$ , and  $\Psi_2^{4,N}$  are constants of motion (i.e their  $u$ -derivatives are zero). This is in agreement with the statement of proposition 2.2 of [13].

## 4 Conserved Quantities.

As it is well known, Bianchi identities for vacuum in the NP formalism split in two groups, those that contain  $D$  and  $\delta$  derivatives (Ba, Bc, Be, Bg according to Stewart's labeling), and those with  $\Delta$  and  $\bar{\delta}$  derivatives (Bb, Bd, Bf & Bh). The first group is used to calculate the coefficients for the leading terms of the components of the Weyl tensor on the initial hypersurface  $\mathcal{N}$ , while the  $u$  dependence comes from the second set because  $\Delta = n^\alpha \nabla_\alpha = \frac{\partial}{\partial u} + Q \frac{\partial}{\partial r} + C^\alpha \frac{\partial}{\partial x^\alpha}$ . So, the Bianchi identities with  $D$  &  $\bar{\delta}$  derivatives are constraint equations on the initial data on the initial hypersurface and on  $\mathfrak{I}^-$  (or  $\mathfrak{I}^+$ ), while the identities with  $\Delta$  &  $\delta$  are the evolution equations for the components of the Weyl tensor. Hence it is reasonable that if we want to find conserved quantities formed from the initial data we have to dig into the expansions for the Bianchi identity that contains  $\Psi_0$  and  $\Delta$  derivatives, namely (Bb). In what follows we will be integrating over the unit sphere  $S^2$ , so it will be useful to rewrite the  $\delta$  derivatives in terms of the eth operator  $\eth$  and its conjugate. For this purpose the relations

$$\eth\eta = \delta\eta + s(\bar{\alpha} - \beta)\eta, \quad (35)$$

$$\bar{\eth}\eta = \bar{\delta}\eta - s(\alpha - \bar{\beta})\eta, \quad (36)$$

will be most useful ( $\eta$  is a quantity of spin weight  $s$ ).

Following Newman & Penrose [5], we consider the  $r^{-6}$ coefficients for the expansion of Bianchi identity (Bb). Directly from the expansion (equation (88) of the appendix A) we get:

$$\begin{aligned} & \dot{\Psi}_0^6 - \frac{1}{2}\Psi_0^{5'} + 2\Psi_0^5 - \frac{1}{2}\left(\Psi_2^3 + \bar{\Psi}_2^3\right)\Psi_0^{4'} + 2\bar{\Psi}_2^3\Psi_0^4 - \bar{\partial}\Psi_1^5 + \sigma_{2,0}\bar{\partial}\Psi_1^4 \\ = & \bar{\partial}\bar{\sigma}_{2,0}\bar{\partial}\Psi_0^4 + \bar{\partial}\sigma_{2,0}\bar{\partial}\Psi_0^4 - 5\Psi_2^3\Psi_0^4 - \bar{\partial}^2\sigma_{2,0}\Psi_0^4 - \sigma_{2,0}\dot{\bar{\sigma}}_{2,0}\Psi_0^4 \\ & + \Psi_2^3\Psi_0^4 - 4\bar{\partial}\sigma_{2,0}\Psi_1^4 + 3\sigma_{2,0}\Psi_2^4. \end{aligned} \quad (37)$$

Dotted quantities are derivatives with respect to  $u$ . From (88) we can read the coefficients for the different powers of  $z$ . In this expression, only the terms  $\dot{\Psi}_0^6$ ,  $2\Psi_0^5$ ,  $\bar{\partial}\Psi_1^5$ ,  $\sigma_{2,0}\bar{\partial}\Psi_1^4$ ,  $\bar{\partial}\sigma_{2,0}\Psi_1^4$ , and  $\sigma_{2,0}\Psi_2^4$  are polynomials of degree  $N$  in  $z = \ln r$  (see table 2). So the coefficient for  $r^{-6} \ln^N r$  is,

$$\dot{\Psi}_0^{6,N} + 2\Psi_0^{5,N} = 3\sigma_{2,0}\Psi_2^{4,N} - 4\bar{\partial}\sigma_{2,0}\Psi_1^{4,N} + \bar{\partial}\Psi_1^{5,N} - \sigma_{2,0}\bar{\partial}\Psi_1^{4,N}. \quad (38)$$

From equations (85) and (89) we can deduce

$$\Psi_1^{5,N} = -\bar{\partial}\Psi_0^{5,N}, \quad (39)$$

$$\Psi_2^{4,N} = -\bar{\partial}\Psi_1^{4,N}. \quad (40)$$

Substitution of the later two expressions into (38) yields:

$$\dot{\Psi}_0^{6,N} = \left(-4\sigma_{2,0}\bar{\partial}\Psi_1^{4,N} - 4\bar{\partial}\sigma_{2,0}\Psi_1^{4,N}\right) + \left(\bar{\partial}\bar{\partial}\Psi_0^{5,N} - 2\Psi_0^{5,N}\right). \quad (41)$$

The terms in the first parenthesis are clearly the  $\bar{\partial}$ -derivative of a product, while in the second parenthesis one we can use the commutation relation for  $\bar{\partial}$  &  $\bar{\partial}$  (equation (99)) yielding:

$$\dot{\Psi}_0^{6,N} = -4\bar{\partial}(\sigma_{2,0}\Psi_1^{4,N}) - \bar{\partial}\bar{\partial}\Psi_0^{5,N}. \quad (42)$$

The first two terms of the right hand side of equation (42) are of the form  $\bar{\partial}$  applied to a quantity of spin weight 3. Multiplying the last two equations by  ${}_2\bar{Y}_{l,m}$  and integrating over  $S^2$  it is possible to apply formula (101) in the case  $s = 2$  and obtain

$$\dot{Q}_m = \int_{S^2} \dot{\Psi}_0^{6,N} ({}_2\bar{Y}_{2,m}) d\omega = 0 \quad (43)$$

for  $m = -2, -1, 0, 1, 2$ . Hence

$$\mathcal{Q}_m = \int_{S^2} \Psi_0^{6,N} ({}_2\bar{Y}_{2,m}) d\omega \quad (44)$$

gives 10 real conserved quantities.

If we try to apply the latter argument to the equations associated to the other powers of  $\ln r$  (and in particular to  $\ln^0 r$ , from which we can construct the NP quantities) we will find that in principle there are several new terms (coming mainly from  $\Psi_0^4$ , see equation (88)) that cannot be written as the  $\bar{\partial}$  derivative of something. The fact that  $\Psi_0^4$  is a polynomial of degree  $N - 1$  in  $\ln r$  happens to be an essential ingredient to obtain the conservation law.

#### 4.1 The conserved quantity of Chruściel et al.

Chruściel et al. found that for an axisymmetric polyhomogeneous space-time which follows the “minimal sequence” the quantity

$$\mathcal{Q}_{XIV} = \int_{S^2} \gamma_{41} \sin^2 \theta d\omega, \quad (45)$$

is conserved. In our treatment, we recover the leading terms of the minimal sequence by setting  $N = 1$ . It is not hard to prove then that for the axisymmetric case (when all the components of the Weyl tensor are real) and  $N = 1$  that

$$\gamma_{41} = -\frac{1}{12} \Psi_0^{6,1}. \quad (46)$$

Now, it can be shown that  $\sin^2 \theta$  is proportional to the spin spherical harmonic  ${}_2\bar{Y}_{2,0}$  [12] so that from (44) we can recover  $\mathcal{Q}_{XIV}$  by putting  $m = 0$ .

### 5 Conclusions.

It has been proven that for a polyhomogeneous space-time in which the leading term of the shear ( $\sigma_2$ ) has no logarithmic terms there are ten conserved quantities. These polyhomogeneous space-times can be interpreted as those where the shear of the outgoing gravitational radiation dominates over the shear of the incoming radiation.

The attempts to interpret the NP quantities have been so far inconclusive (see for example [5], [16]). It also has been suggested that these quantities lack a physical meaning [19]. However most authors agree that the NP quantities reflect in some way the structure of the incoming radiation. This opinion is reinforced with the conserved quantities  $\mathcal{Q}_m$ , because they possess an analogous structure, and because the kind of space-times we are working with admit much stronger incoming gravitational radiation than the asymptotically flat space-times used by Newman and Penrose.

It has to be mentioned that we suspect that for a general polyhomogeneous space-time the quantities  $\mathcal{Q}_m$  cease to be constants, or at least the arguments used here cannot be applied anymore due to the appearance of several new terms that hardly could be rewritten as a  $\bar{\partial}$  derivative. In more physical terms this could be rephrased by saying that the  $\mathcal{Q}_m$  are no longer conserved because the shear of the incoming radiation dominates over that of the outgoing radiation. In a similar way the NP quantities are not constants in a polyhomogeneous setting because the incoming radiation is not of finite duration (see for example [5], [8]). A deeper exploration of these ideas will be the subject of future work.

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## A Polyhomogeneous expansions in the NP formalism.

### A.1 The relationship between $\rho$ , $\sigma$ & $\Psi_0$ for a general polyhomogeneous space-time.

Beginning with equations (21) and (22)

$$\rho'_{i-1} - (i-1) \rho_{i-1} = \sum_{k=1}^{i-1} (\rho_k \rho_{i-k} + \sigma_k \bar{\sigma}_{i-k}), \quad (47)$$

$$\sigma'_{i-1} - (i-1) \sigma_{i-1} = 2 \sum_{k=1}^{i-1} \rho_k \sigma_{i-k} + \Psi_0^i, \quad (48)$$

we find for  $i = 1$  that

$$\Psi_0^1 = 0; \quad (49)$$

for  $i = 2$ ,

$$\rho_1 = \rho_{1,0} = -1, \quad (50)$$

$$\sigma_1 = 0, \quad (51)$$

$$\Psi_0^2 = 0. \quad (52)$$

For  $i = 3$  we find that

$$\rho'_2 = 0, \quad (53)$$

$$\sigma'_2 = \Psi_0^3. \quad (54)$$

So  $\rho_2$  is a polynomial of degree 0 in  $\ln r$ . Using the remaining freedom in the definition of  $r$  (recall that  $r$  is an affine parameter) we can redefine  $r$  such that  $\rho_2 = 0$  (see for example [3]). From the other equation we read

$$\Psi_0^{3,N} = 0, \quad (55)$$

$$\Psi_0^{3,j} = (j+1)\sigma_{2,j+1}. \quad (56)$$

Hence we can think of  $\Psi_0^{3,j}$   $j = 0 \dots N-1$  as a contribution to the shear from the incoming radiation using Szekeres interpretation of the components of the Weyl tensor ( $\Psi_0$  can be regarded as an incoming transverse wave)[18].

For  $i = 4$  we obtain,

$$\sigma_{2,N} = \sigma_{2,N-1} = \dots = \sigma_{2,\|N/2\|} = 0, \quad (57)$$

$$\Psi_0^{3,N-1} = \dots = \Psi_0^{3,\|N/2\|-1} = 0, \quad (58)$$

$$\rho_{3,N} = 0, \quad (59)$$

$$-\rho_{3,j} + (j+1)\rho_{3,j+1} = \sum_{k=0}^j \sigma_{2,k} \bar{\sigma}_{2,j-k}, \quad (60)$$

$$\sigma_{3,N} = -\Psi_0^{4,N}, \quad (61)$$

$$-\sigma_{3,j} + (j+1)\sigma_{3,j+1} = \Psi_0^{4,j}. \quad (62)$$

for  $j = 0 \dots N-1$ . Using the relations for  $i = 6$  it is possible to find further bounds on  $\sigma_3$  and  $\Psi_0^4$ .

## A.2 Orders of the leading terms for a general polyhomogeneous space-time.

Using a similar technique we can set bounds for the orders of the “logarithmic polynomials” of the leading terms for the remaining spin coefficients, components of the Weyl tensor and tetrad functions. The degrees of the polynomials are listed as a table. The - means that such a term does not appear in the expansion.

	1	$r^{-1}$	$r^{-2}$	$r^{-3}$	$r^{-4}$	$r^{-5}$	$r^{-6}$
$\Psi_0$	-	-	-	$\ N/2\  - 1$	$N - 1$	$N$	$N$
$\Psi_1$	-	-	-	$\ N/2\  - 1$	$N$	...	...
$\Psi_2$	-	-	-	$\ N/2\ $	$N$	...	...
$\Psi_3$	-	-	0	$\ N/2\ $	...	...	...
$\Psi_4$	-	0	0	...	...	...	...
$\rho$	-	0	0	$N - 1$	$N$	...	...
$\sigma$	-	-	$\ N/2\ $	$N$	$N$	...	...
$\alpha$	-	0	$\ N/2\  + 1$	$N$	...	...	...
$\beta$	-	0	$\ N/2\ $	$N$	...	...	...
$\tau$	-	-	$\ N/2\  + 1$	$N$	...	...	...
$\pi$	-	-	$\ N/2\  + 1$	$N$	...	...	...
$\gamma$	-	0	$\ N/2\  + 1$	...	...	...	...
$\lambda$	-	0	$\ N/2\  + 1$	...	...	...	...
$\mu$	-	0	$\ N/2\  + 1$	...	...	...	...
$v$	-	0	...	...	...	...	...
$\xi^\alpha$	-	0	$\ N/2\ $	$N$	...	...	...
$Q$	0	$\ N/2\  + 1$	...	...	...	...	...
$C^\alpha$	-	-	$\ N/2\  + 1$	...	...	...	...

(Table 1)

### A.3 Orders for the leading terms for polyhomogeneous space-times such that $\sigma_2 = \sigma_{2,0}(\theta, \varphi)$ .

The restriction  $\sigma_2 = \sigma_{2,0}(\theta, \varphi)$  puts lower bounds on the orders of the “logarithmic polynomials”. These are listed in table 2.

	1	$r^{-1}$	$r^{-2}$	$r^{-3}$	$r^{-4}$	$r^{-5}$	$r^{-6}$
$\Psi_0$	-	-	-	-	$N - 1$	$N$	$N$
$\Psi_1$	-	-	-	-	$N$	...	...
$\Psi_2$	-	-	-	0	$N$	...	...
$\Psi_3$	-	-	0	0	...	...	...
$\Psi_4$	-	0	0	0	...	...	...
$\rho$	-	0	0	0	$N - 1$	$N$	...
$\sigma$	-	-	0	$N - 1$	$N$	...	...
$\alpha$	-	0	0	$N$	...	...	...
$\beta$	-	0	0	$N$	...	...	...
$\tau$	-	-	0	$N$	...	...	...
$\pi$	-	-	0	$N$	...	...	...
$\gamma$	-	-	0	...	...	...	...
$\lambda$	-	0	0	...	...	...	...
$\mu$	-	0	0	...	...	...	...
$v$	-	0	...	...	...	...	...
$\xi^\alpha$	-	0	0	$N - 1$	...	...	...
$Q$	0	0	...	...	...	...	...
$C^\alpha$	-	-	0	...	...	...	...

(Table2)

So that the leading terms are,

$$\Psi_0 = \left( \sum_{k=0}^{N-1} \Psi_0^{4,k} \ln^k r \right) r^{-4} + \dots , \quad (63)$$

$$\Psi_1 = \left( \sum_{k=0}^N \Psi_1^{4,k} \ln^k r \right) r^{-4} + \dots , \quad (64)$$

$$\Psi_2 = (\Psi_2^{3,0}) r^{-3} + \dots , \quad (65)$$

$$\Psi_3 = -\bar{\partial} \dot{\bar{\sigma}}_{2,0} r^{-2} + \dots , \quad (66)$$

$$\Psi_4 = -\ddot{\bar{\sigma}}_{2,0} r^{-1} + \dots , \quad (67)$$

$$\rho = -r^{-1} - (\sigma_{2,0} \bar{\sigma}_{2,0}) r^{-3} + \dots , \quad (68)$$

$$\sigma = \sigma_{2,0} r^{-2} + \dots , \quad (69)$$

$$\alpha = \alpha_{1,0} r^{-1} + (\bar{\partial} \bar{\sigma}_{2,0} + \bar{\alpha}_{1,0} \bar{\sigma}_{2,0}) r^{-2} + \dots , \quad (70)$$

$$\beta = -\bar{\alpha}_{1,0} r^{-1} - \alpha_{1,0} \sigma_{2,0} r^{-2} + \dots , \quad (71)$$

$$\tau = \bar{\partial} \sigma_{2,0} r^{-2} + \dots , \quad (72)$$

$$\pi = \bar{\partial} \bar{\sigma}_{2,0} r^{-2} + \dots , \quad (73)$$

$$\gamma = - \left( \alpha_{1,0} \bar{\partial} \sigma_{2,0} - \bar{\alpha}_{1,0} \bar{\partial} \bar{\sigma}_{2,0} + \frac{1}{2} \Psi_2^{3,0} \right) r^{-2} + \dots , \quad (74)$$

$$\mu = -\frac{1}{2} r^{-1} + \left( \bar{\partial}^2 \bar{\sigma}_{2,0} + \sigma_{2,0} \dot{\bar{\sigma}}_{2,0} + \Psi_2^{3,0} \right) r^{-2} + \dots , \quad (75)$$

$$\lambda = \dot{\bar{\sigma}}_{2,0} r^{-1} - \left( \bar{\partial} \bar{\partial} \bar{\sigma}_{2,0} - \frac{1}{2} \bar{\sigma}_{2,0} \right) r^{-2} + \dots , \quad (76)$$

$$v = \frac{1}{2} \bar{\partial} \left( \Psi_2^3 + \bar{\Psi}_2^3 \right) r^{-2} + \dots \quad (77)$$

$$\text{and } \kappa = \epsilon = 0, \quad (78)$$

$$\text{where } \alpha_{1,0} = -\frac{1}{2\sqrt{2}} \cot \theta. \quad (79)$$

## A.4 The tetrad functions.

The tetrad functions are calculated from the frame equations that are obtained from applying the commutators of the different derivatives ( $D, \Delta, \delta, \bar{\delta}$ ) to  $u, r$ , and to  $x^i$ . The results are,

$$\xi^\alpha = \xi_{1,0}^\alpha r^{-1} - \sigma_{2,0} \bar{\xi}_{1,0}^\alpha r^{-2} + \dots , \quad (80)$$

$$C^i = - \left( \bar{\partial} \sigma_{2,0} \xi^{0i} + \bar{\partial} \sigma_{2,0} \bar{\xi}^{0i} \right) r^{-2} + \dots , \quad (81)$$

$$Q = -\frac{1}{2} - \frac{1}{2} \left( \Psi_2^3 + \bar{\Psi}_2^3 \right) r^{-1} + \dots ,$$

where

$$\xi^{0\theta} = \frac{1}{\sqrt{2}}, \quad (82)$$

$$\xi^{0\varphi} = -\frac{i}{\sqrt{2}} \csc \theta. \quad (83)$$

## A.5 Relations obtained from the Bianchi identities when $\sigma_2 = \sigma_{2,0}(\theta, \varphi)$ .

### A.5.1 (Ba).

$$\Psi_1^{4'} = \bar{\partial} \Psi_0^4, \quad (84)$$

$$\Psi_1^{5'} - \Psi_1^5 = \bar{\partial} \Psi_0^5 + (\bar{\beta}_2 - 3\alpha_2) \Psi_0^4. \quad (85)$$

### A.5.2 (Bb).

$$\dot{\Psi}_0^{4,N-1} = 0 , \quad (86)$$

$$\dot{\Psi}_0^{5,N} = \bar{\partial} \Psi_1^{4,N}, \quad (87)$$

$$\begin{aligned}
& \dot{\Psi}_0^6 - \frac{1}{2} \Psi_0^{5'} + 2\Psi_0^5 - \frac{1}{2} \left( \Psi_2^3 + \bar{\Psi}_2^3 \right) \Psi_0^{4'} + 2\bar{\Psi}_2^3 \Psi_0^4 - \partial\Psi_1^5 + \sigma_{2,0} \bar{\partial}\Psi_1^4 \\
= & \partial\bar{\sigma}_{2,0} \partial\Psi_0^4 + \bar{\partial}\sigma_{2,0} \bar{\partial}\Psi_0^4 - 5\Psi_2^3 \Psi_0^4 - \partial^2 \sigma_{2,0} \Psi_0^4 - \sigma_{2,0} \dot{\bar{\sigma}}_{2,0} \Psi_0^4 \\
& + \Psi_2^3 \Psi_0^4 - 4\bar{\partial}\sigma_{2,0} \Psi_1^4 + 3\sigma_{2,0} \Psi_2^4. \tag{88}
\end{aligned}$$

**A.5.3 (Bc).**

$$\Psi_2^{4'} - \Psi_2^4 = \dot{\bar{\sigma}}_{2,0} \Psi_0^4 + \bar{\partial}\Psi_1^4. \tag{89}$$

**A.5.4 (Bd).**

$$\dot{\Psi}_1^{4,N} = 0, \tag{90}$$

$$\dot{\Psi}_1^{5,N} = \partial\Psi_2^{4,N} - \Psi_1^{4,N}. \tag{91}$$

**A.5.5 (Be).**

$$\Psi_3^{3,0} = -\bar{\partial}\Psi_2^{3,0}. \tag{92}$$

**A.5.6 (Bf).**

$$\dot{\Psi}_2^{4,N} = 0. \tag{93}$$

**A.5.7 (Bg).**

$$\Psi_4^{2,0} = -\bar{\partial}\Psi_3^{2,0}. \tag{94}$$

## B Properties of $\mathfrak{D}$ & $\bar{\mathfrak{D}}$ .

The properties of the differential operator  $\mathfrak{D}$  relevant for the present article are presented here as a quick reference. For proofs and a more extended discussion refer to [5], [7], [9].

A quantity  $\eta$  is said to have spin weight  $s$  if under the tetrad transformation

$$m^{\mu'} = e^{i\psi} m^\mu, \tag{95}$$

it transforms as

$$\eta' = e^{si\psi} \eta. \quad (96)$$

For example  $\sigma$  has spin weight 2, and  $\Psi_k$  has spin weight  $2 - k$ . Let now  $\eta$  be a quantity of spin weight  $s$  defined on the sphere. We define the operators  $\eth$  and  $\bar{\eth}$  by

$$\eth \eta = -(\sin \theta)^s \left\{ \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \theta} \right\} \{ (\sin \theta)^{-s} \eta \}, \quad (97)$$

$$\bar{\eth} \eta = (\sin \theta)^{-s} \left\{ \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \theta} \right\} \{ (\sin \theta)^s \eta \}. \quad (98)$$

$\eth \eta$  has spin weight  $s + 1$ , and  $\bar{\eth} \eta$  has spin weight  $s - 1$ . Their commutator is:

$$(\eth \bar{\eth} - \bar{\eth} \eth) \eta = 2s\eta. \quad (99)$$

The spin  $s$  spherical harmonics are defined as ( $s$  integer),

$${}_s Y_{l,m} = \begin{cases} \left[ \frac{(l-s)!}{(l+s)!} \right]^{1/2} \eth^s Y_{l,m} & \text{for } 0 \leq s \leq l, \\ (-1)^s \left[ \frac{(l+s)!}{(l-s)!} \right] \bar{\eth}^{-s} Y_{l,m} & \text{for } -l \leq s \leq 0. \end{cases} \quad (100)$$

And finally we have that if  $\zeta$  is of spin weight  $l + 1$  then,

$$\int_{S^2} ({}_s \bar{Y}_{l,m}) \bar{\eth}^{l-s+1} \zeta d\omega = 0. \quad (101)$$

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